Poisson Geometry and Normal Forms: A Guided Tour through Examples

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From Poisson Geometry to Quantum Fields on Noncommutative Spaces, Würzburg Autumn School

Lectures 4 and 5

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Lectures 4 and 5

Symplectic Geometry	Poisson Geometry
ω	Π
$\iota_{X_f}\omega = -df$	$X_f := \Pi(df, \cdot)$
one symplectic leaf	a symplectic foliation
Darboux theorem	Weinstein's splitting theorem
$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$	$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_i} + \sum_{kl} \phi_{kl}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l}$
	$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_i} + \sum_{rs} c_{rs}^{k} x_k \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l}$
$L_X \omega = 0$	$L_X \Pi = 0$
$H^1_{DR}(M) = \frac{\text{symplectic v.f}}{\text{Hamiltonian v.f}}$	$? = \frac{\text{Poisson v.f}}{\text{Hamiltonian v.f}}$
$H^k_{DR}(M)$ (cochains $\Omega^k(M)$)	$?:= H^k_{\Pi}(M)$ (cochains $\mathfrak{X}^m(M)$)
Arnold-Liouville theorem	Action-angle coordinates

Plan for today

- Poisson cohomology computation kit.

- Integrable systems on Poisson manifolds (Topology).

- Integrable systems on Poisson manifolds (Geometry and normal forms). The case of b-Poisson manifolds.
- Applications.

Schouten Bracket of vector fields in local coordinates

- Case of vector fields,
  $A = \sum_i a_i \frac{\partial}{\partial x_i}$ and $B = \sum_i b_i \frac{\partial}{\partial x_i}$. Then

$$[A, B] = \sum_i a_i (\sum_j \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j}) - \sum_i b_i (\sum_j \frac{\partial a_j}{\partial x_i} \frac{\partial}{\partial x_j})$$

- Re-denoting $\frac{\partial}{\partial x_i}$ as ζ_i **("odd coordinates ")**.
  Then $A = \sum_i a_i \zeta_i$ and $B = \sum_i b_i \zeta_i$ and $\zeta_i \zeta_j = -\zeta_j \zeta_i$ Now we can reinterpret the bracket as,

$$[A, B] = \sum_i \frac{\partial A}{\partial \zeta_i} \frac{\partial B}{\partial x_i} - \sum_i \frac{\partial B}{\partial \zeta_i} \frac{\partial A}{\partial x_i}$$

Schouten Bracket of multivector fields in local coordinates

We reproduce the same scheme for the case of multivector fields.

$$[A, B] = \sum_i \frac{\partial A}{\partial \zeta_i} \frac{\partial B}{\partial x_i} - (-1)^{(a-1)(b-1)} \sum_i \frac{\partial B}{\partial \zeta_i} \frac{\partial A}{\partial x_i}$$

is a $(a+b-1)$-vector field.
where

$$A = \sum_{i_1 < \cdots < i_a} A_{i_1,\ldots,i_a} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_a}} = \sum_{i_1 < \cdots < i_a} A_{i_1,\ldots,i_a} \zeta_{i_1} \ldots \zeta_{i_a}$$

and

$$B = \sum_{i_1 < \cdots < i_b} B_{i_1,\ldots,i_b} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_b}} = \sum_{i_1 < \cdots < i_b} B_{i_1,\ldots,i_b} \zeta_{i_1} \ldots \zeta_{i_b}$$

with $\frac{\partial(\zeta_{i_1} \ldots \zeta_{i_p})}{\partial \zeta_{i_k}} := (-1)^{(p-k)} \eta_{i_1} \ldots \widehat{\eta}_{i_k} \eta_{i_{p-1}}$

Theorem (Schouten-Nijenhuis)

The bracket defined by this formula satisfies,

Graded anti-commutativity $[A, B] = -(-1)^{(a-1)(b-1)}[B, A]$.

Graded Leibniz rule

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(a-1)b} B \wedge [A, C]$$

Graded Jacobi identity

$$(-1)^{(a-1)(c-1)}[A, [B, C]] + (-1)^{(b-1)(a-1)}[B, [C, A]] + (-1)^{(c-1)(b-1)}[C, [A, B]] = 0$$

If X is a vector field then, $[X, B] = L_X B$.

Poisson cohomology computation kit

- Space of cochains $\mathfrak{X}^m(M)$.
- Differential $d_\Pi(A) := [\Pi, A]$.
- Poisson cohomology

$$H^k_\Pi(M) := \frac{\ker d_\Pi : \mathfrak{X}^k(M) \longrightarrow \mathfrak{X}^{k+1}(M)}{\mathrm{Im} d_\Pi : \mathfrak{X}^{k-1}(M) \longrightarrow \mathfrak{X}^k(M)}$$

- Computation is difficult. It can be infinite-dimensional. Tools: Mayer-Vietoris, spectral sequences.
- Particular cases: (M, Π) symplectic $H^k_\Pi(M) \cong H^k_{DR}(M)$.
- (M, Π) b-Poisson, $H^k_\Pi(M) \cong H^k_{DR}(M) \oplus H^{k-1}_{DR}(Z)$.

Poisson cohomology computation kit

- Hamiltonian vector fields $X_f = -[\Pi, f]$ (1-coboundary).
- Poisson vector fields $[\Pi, X] = -L_X \Pi = 0$ (1-cocycle).
- Poisson structures $[\Pi, \Pi] = 0$ (2-cocycle).
- Compatible Poisson structures $[\Pi_1, \Pi_2] = 0$ (2-cocycle).
- $$H^1_\Pi = \frac{\text{Poisson vector fields}}{\text{Hamiltonian vector fields}}.$$

Example 5: Cauchy-Riemann equations and Hamilton's equations

- Take a holomorphic function on $F : \mathbb{C}^2 \longrightarrow \mathbb{C}$ decompose it as $F = G + iH$ with $G, H : \mathbb{R}^4 \longrightarrow \mathbb{R}$.

  Cauchy-Riemann equations for F in coordinates $z_j = x_j + iy_j$, $j = 1, 2$

$$\frac{\partial G}{\partial x_i} = \frac{\partial H}{\partial y_i}, \quad \frac{\partial G}{\partial y_i} = -\frac{\partial H}{\partial x_i}$$

- Reinterpret these equations as the equality

$$\{G, \cdot\}_0 = \{H, \cdot\}_1 \quad \{H, \cdot\}_0 = -\{G, \cdot\}_1$$

  with $\{\cdot, \cdot\}_j$ the Poisson brackets associated to the real and imaginary part of the symplectic form $\omega = dz_1 \wedge dz_2$ $(\omega = \omega_0 + i\omega_1)$.
- Check $\{G, H\}_0 = 0$ and $\{H, G\}_1 = 0$ (integrable system).

Example 2: Determinants in $\mathbb{R}^3$ (Exercise 12)

- Dynamics: Given two functions $H, K \in \mathcal{C}^\infty(\mathbb{R}^3)$. Consider the system of differential equations:

$$(\dot{x}, \dot{y}, \dot{z}) = dH \wedge dK \qquad (1)$$

  H and K are constants of motion (the flow lies on $H = cte.$ and $K = cte.$)

- Geometry: Consider the brackets,

$$\{f, g\}_H := \det(df, dg, dH) \quad \{f, g\}_K := \det(df, dg, dK)$$

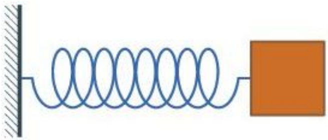
  They are antisymmetric and satisfy Jacobi,
  $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$
  The flow of the vector field

$$\{K, \cdot\}_H := \det(dK, \cdot, dH)$$

  and $\{-H, \cdot\}_K$ is given by the differential equation (1) and

$$\{H, K\}_H = 0, \quad \{H, K\}_K = 0$$

Example 4: Coupling two simple harmonic oscillators



The phase space is $(T^*(\mathbb{R}^2), \omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$. H is the sum of potential and kinetic energy,

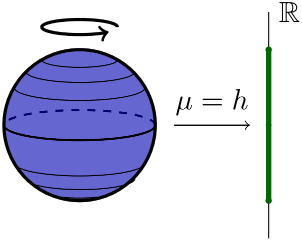
$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(x_1^2 + x_2^2)$$

$H = h$ is a sphere S^3. We have rotational symmetry on this sphere $\rightsquigarrow$ the angular momentum is a constant of motion, $L = x_1 y_2 - x_2 y_1$, $X_L = (-x_2, x_1, -y_2, y_1)$ and

$$X_L(H) = \{L, H\} = 0.$$

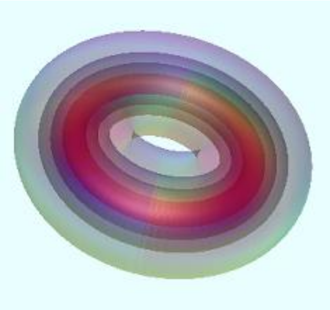
Topology of integrable systems (Symplectic case)

An integrable system on a surface.



The invariant submanifolds are tori (Liouville tori)

Liouville-Mineur-Arnold theorem (Symplectic manifolds)



The orbits of an integrable system in a neighbourhood of a compact orbit are tori. In action-angle coordinates (p_i, θ_i) the foliation is given by the fibration $\{p_i = c_i\}$ and the symplectic structure is Darboux $\omega = \sum_{i=1}^{n} dp_i \wedge d\theta_i$.

The characters of the day

Joseph Liouville proved the existence of invariant manifolds.

Figure: Joseph Liouville, Henri Mineur, Duistermaat and Arnold

Henri Mineur gave a explicit formula for action coordinates: $p_i = \int_{\gamma_i} \alpha$ where γ_i is one of the cycles of the Liouville torus and α is a Liouville 1-form for the symplectic structure $(\omega = d\alpha)$.

We will follow the proof by Duistermaat and apply it to Poisson manifolds.

What is an integrable system on a Poisson manifold?

Let (M, Π) be a Poisson manifold of (maximal) rank $2r$ and of dimension n. An s-tuplet of functions $\mathbf{F} = (\mathbf{f_1}, \dots, \mathbf{f_s})$ on M is said to define a Liouville integrable system on (M, Π) if

1. $f_1, \dots, f_s$ are independent ($df_1 \wedge \cdots \wedge df_s \neq 0$).
2. $f_1, \dots, f_s$ are pairwise in involution
3. $r + s = n$

Viewed as a map, $\mathbf{F} : \mathbf{M} \to \mathbf{R^s}$ is called the *moment map* of (M, Π).

A Darboux-Carathéodory theorem in the Poisson context

Theorem (Laurent, M., Vanhaecke)

Let $p_1, \dots, p_r$ be r functions in involution and whose Hamiltonian vector fields are linearly independent at a point $m \in (M, \Pi)$. There exist locally functions $q_1, \dots, q_r, z_1, \dots, z_{n-2r}$, such that

1. *The n functions $(p_1, q_1, \dots, p_r, q_r, z_1, \dots, z_{n-2r})$ form a system of coordinates on U, centered at m;*
2. *The Poisson structure Π is given on U by*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} + \sum_{i,j=1}^{n-2r} g_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{1}$$

Coffee time

An action-angle theorem for Poisson manifolds

Case of regular orbits

We assume that:

1. The mapping $\mathcal{F} = (f_1, \ldots, f_s)$ defines an integrable system on the Poisson manifold (M, Π) of dimension n and (maximal) rank $2r$.
2. Suppose that $m \in M$ is a point such that it is regular for the integrable system and the Poisson structure.
3. Assume further than the integral manifold $\mathcal{F}_m$ of the foliation $X_{f_1}, \ldots X_{f_s}$ through m is compact (Liouville torus).

An action-angle theorem for Poisson manifolds

Theorem (Laurent, M., Vanhaecke)

There exist $\mathbf{R}$*-valued smooth functions* $(p_1, \ldots, p_s)$ *and* $\mathbf{R}/\mathbf{Z}$*-valued smooth functions* $(\theta_1, \ldots, \theta_r)$*, defined in a neighborhood of* $\mathcal{F}_m$ *such that*

1. *The functions* $(\theta_1, \ldots, \theta_r, p_1, \ldots, p_s)$ *define a diffeomorphism* $U \simeq \mathbf{T}^r \times B^s$;
2. *The Poisson structure can be written in terms of these coordinates as*

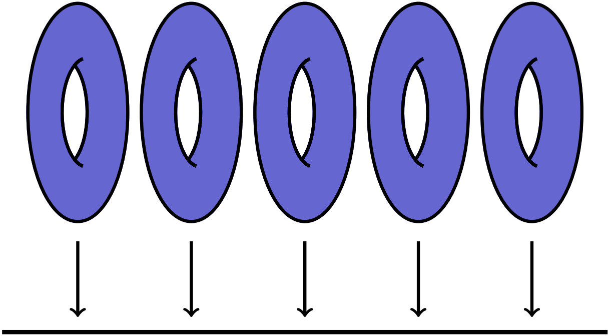
$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial \theta_i},$$

in particular the functions $p_{r+1}, \ldots, p_s$ *are locally Casimirs of* Π;

3. *The leaves of the surjective submersion* $\mathcal{F} = (f_1, \ldots, f_s)$ *are given by the projection onto the second component* $\mathbf{T}^r \times B^s$*, in particular, the functions* $p_1, \ldots, p_s$ *depend only on the functions* $f_1, \ldots, f_s$.

The Poisson proof

- Step 1: Topology of the foliation. The fibration in a neighbourhood of a compact connected fiber is a trivial fibration by compact fibers. The fibers are tori.



The Poisson proof

- Step 2: Hamiltonian action: We recover a $\mathbb{T}^r$-action tangent to the leaves of the foliation. This implies a process of uniformization of periods.

$$\begin{array}{rcccl} \Phi & : & \mathbf{R}^r \times (\mathbf{T}^r \times B^s) & \to & \mathbf{T}^r \times B^s \\ & & ((t_1, \ldots, t_r), m) & \mapsto & \Phi^{(1)}_{t_1} \circ \cdots \circ \Phi^{(r)}_{t_r}(m). \end{array} \tag{2}$$

- Step 3: We prove that this action is Poisson (if Y is a complete vector field of period 1 and P is a bivector field for which $\mathcal{L}_Y^2 P = 0$, then $\mathcal{L}_Y P = 0$).
- Step 4: Finally we use the Poisson Cohomology of the manifold and to check that the action is Hamiltonian.
- Step 5: To construct action-angle coordinates we use Darboux-Carathéodory and the constructed Hamiltonian action of $\mathbb{T}^n$ to drag normal forms from a neighbourhood of a point to a neighbourhood of a fiber.

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What is a non-commutative integrable system on a Poisson manifold?

Definition

Let (M, Π) be a Poisson manifold of dimension n. An s-uplet of functions $\mathcal{F} = (f_1, \ldots, f_s)$ is said to be a *non-commutative integrable system of rank* r on (M, Π) if

(1) $f_1, \ldots, f_s$ are independent;

(2) The functions $f_1, \ldots, f_r$ are in involution with the functions $f_1, \ldots, f_s$;

(3) $r + s = n$;

(4) The Hamiltonian vector fields of the functions $f_1, \ldots, f_r$ are linearly independent at some point of M.

Notice that $2r \leq \operatorname{Rk} \Pi$, as a consequence of (4).

Remark: The mapping $\mathcal{F} = (f_1, \ldots, f_s)$ is a Poisson map on $\mathbb{R}^s$ with $\mathbb{R}^s$ endowed with a non-vanishing Poisson structure.

An action-angle theorem for non-commutative systems

Theorem (Laurent, M., Vanhaecke)

Suppose that $\mathcal{F}_m$ is a regular Liouville torus. Then there exist semilocally $\mathbf{R}$-valued smooth functions $(p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ and $\mathbf{R}/\mathbf{Z}$-valued smooth functions $(\theta_1, \ldots, \theta_r)$ such that,

1. *The functions $(\theta_1, \ldots, \theta_r, p_1, \ldots, p_r, z_1, \ldots, z_{s-r})$ define a diffeomorphism $U \simeq \mathbf{T}^r \times B^s$;*
2. *The Poisson structure can be written in terms of these coordinates as*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial \theta_i} + \sum_{k,l=1}^{s-r} \phi_{k,l}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l};$$

3. *The leaves of the surjective submersion $\mathcal{F} = (f_1, \ldots, f_s)$ are given by the projection onto the second component $\mathbf{T}^r \times B^s$, in particular, the functions $p_1, \ldots, p_r, z_1, \ldots, z_{s-r}$ depend on the functions $f_1, \ldots, f_s$ only.*

The restricted 3-body problem

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler's laws for the 2-body problem.

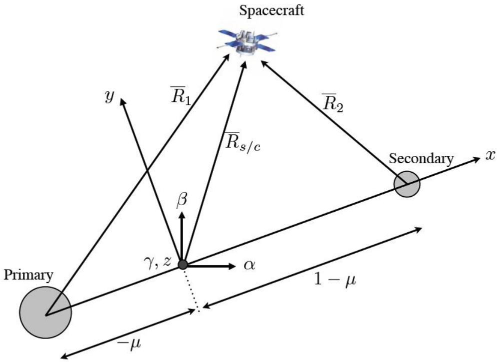

Figure: Circular 3-body problem

Planar restricted 3-body problem[1]

- The time-dependent self-potential of the small body is $U(q,t) = \frac{1-\mu}{|q-q_1|} + \frac{\mu}{|q-q_2|}$, with $q_1 = q_1(t)$ the position of the planet with mass $1-\mu$ at time t and $q_2 = q_2(t)$ the position of the one with mass μ.
- The Hamiltonian of the system is $H(q,p,t) = p^2/2 - U(q,t), \quad (q,p) \in \mathbf{R}^2 \times \mathbf{R}^2$, where $p = \dot{q}$ is the momentum of the planet.
- Consider the canonical change $(X, Y, P_X, P_Y) \mapsto (r, \alpha, P_r =: y, P_\alpha =: G)$.
- Introduce **McGehee coordinates** (x, α, y, G), where $r = \frac{2}{x^2}, \quad x \in \mathbf{R}^+$, can be then extended to infinity ($x = 0$).
- The symplectic structure becomes a singular object $\omega = -\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG$. for $x > 0$
- The integrable 2-body problem for $\mu = 0$ is integrable with respect to the singular ω.

[1] Thanks to Amadeu Delshams for this example.

Definition (b-integrable system)

A set of b-functions $f_1, \dots, f_n$ on (M^{2n}, ω) such that

- $f_1, \dots, f_n$ Poisson commute
- $df_1 \wedge \dots \wedge df_n \neq 0$ as a section of $\Lambda^n({}^bT^*(M))$ on a dense subset of M and on a dense subset of Z

Theorem (Kiesenhofer,M., Scott)

Let (M,Π) be a b-Poisson manifold with critical hypersurface Z defined by $t=0$. Let $f_1,\dots,f_{n-1},f_n=\log|t|$ be a b-integrable system on it. Then in a neighbourhood of a Liouville torus $_m$ there exist coordinates $(\theta_1,\dots,\theta_n,a_1,\dots,a_n):U\to\mathbf{T}^n\times B^n$ such that

$$\Pi|_U=\sum_{i=1}^{n-1}\frac{\partial}{\partial a_i}\wedge\frac{\partial}{\partial\theta_i}+ct\frac{\partial}{\partial t}\wedge\frac{\partial}{\partial\theta_n}, \tag{3}$$

where the coordinates $a_1,\dots,a_{n-1}$ depend only on $f_1,\dots,f_{n-1}$ and c is the modular period of the component of Z containing m.

KAM

Perturbation theory KAM on manifolds with boundary and b-manifolds. (see Anna's poster).