Poisson Geometry and Normal Forms: A Guided Tour through Examples

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From Poisson Geometry to Quantum Fields on Noncommutative Spaces, Würzburg Autumn School

Lecture 3

Symplectic Geometry	Poisson Geometry
ω	Π
$\iota_{X_f}\omega = -df$	$X_f := \Pi(df, \cdot)$
one symplectic leaf	a symplectic foliation
Darboux theorem	Weinstein's splitting theorem
$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$	$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_i} + \sum_{kl} \phi_{kl}(z) \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_l}$
$L_X\omega = 0$	$L_X\Pi = 0$
$H^1_{DR}(M) = \frac{\text{symplectic v.f}}{\text{Hamiltonian v.f}}$	$? = \frac{\text{Poisson v.f}}{\text{Hamiltonian v.f}}$
$H^k_{DR}(M)$ (cochains $\Omega^k(M)$)	$? := H^k_{\Pi}(M)$ (cochains $\mathfrak{X}^m(M)$)

Plan for today

- Weinstein's splitting theorem and normal form theorems.

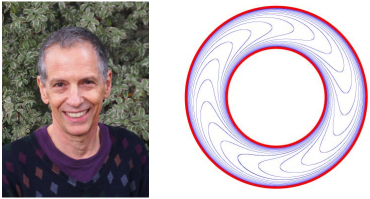

Figure: Alan Weinstein and Reeb foliation

- Poisson cohomology. Some computations.
- Compatible Poisson structures and commuting first integrals.

Schouten Bracket of vector fields in local coordinates

- Case of vector fields,
  $A = \sum_i a_i \frac{\partial}{\partial x_i}$ and $A = \sum_i b_i \frac{\partial}{\partial x_i}$. Then

$$[A, B] = \sum_i a_i (\sum_j \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j}) - \sum_i b_i (\sum_j \frac{\partial a_j}{\partial x_i} \frac{\partial}{\partial x_j})$$

- Re-denoting $\frac{\partial}{\partial x_i}$ as ζ_i **("odd coordinates ")**.
  Then $A = \sum_i a_i \zeta_i$ and $B = \sum_i b_i \zeta_i$ and $\zeta_i \zeta_j = -\zeta_j \zeta_i$ Now we can reinterpret the bracket as,

$$[A, B] = \sum_i \frac{\partial A}{\partial \zeta_i} \frac{\partial B}{\partial x_i} - \sum_i \frac{\partial B}{\partial \zeta_i} \frac{\partial A}{\partial x_i}$$

Schouten Bracket of multivector fields in local coordinates

We reproduce the same scheme for the case of multivector fields.

$$[A, B] = \sum_i \frac{\partial A}{\partial \zeta_i}\frac{\partial B}{\partial x_i} - (-1)^{(a-1)(b-1)} \sum_i \frac{\partial B}{\partial \zeta_i}\frac{\partial A}{\partial x_i}$$

is a $(a+b-1)$-vector field.
where

$$A = \sum_{i_1 < \cdots < i_a} A_{i_1,\ldots,i_a} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_a}} = \sum_{i_1 < \cdots < i_a} A_{i_1,\ldots,i_a} \zeta_{i_1} \ldots \zeta_{i_a}$$

and

$$B = \sum_{i_1 < \cdots < i_b} B_{i_1,\ldots,i_b} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_b}} = \sum_{i_1 < \cdots < i_b} B_{i_1,\ldots,i_b} \zeta_{i_1} \ldots \zeta_{i_b}$$

with $\frac{\partial(\zeta_{i_1} \cdots \zeta_{i_p})}{\partial \zeta_{i_k}} := (-1)^{(p-k)} \eta_{i_1} \ldots \widehat{\eta}_{i_k} \eta_{i_{p-1}}$

Theorem (Schouten-Nijenhuis)

The bracket defined by this formula satisfies,

Graded anti-commutativity $[A, B] = -(-1)^{(a-1)(b-1)}[B, A]$.

Graded Leibniz rule

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(a-1)b} B \wedge [A, C]$$

Graded Jacobi identity

$$(-1)^{(a-1)(c-1)}[A, [B, C]] + (-1)^{(b-1)(a-1)}[B, [C, A]] + (-1)^{(c-1)(b-1)}[C, [A, B]] = 0$$

If X *is a vector field then,* $[X, B] = L_X B$.

Example 5: Cauchy-Riemann equations and Hamilton's equations

- Take a holomorphic function on $F : \mathbb{C}^2 \longrightarrow \mathbb{C}$ decompose it as $F = G + iH$ with $G, H : \mathbb{R}^4 \longrightarrow \mathbb{R}$.

  Cauchy-Riemann equations for F in coordinates $z_j = x_j + iy_j$, $j = 1, 2$

$$\frac{\partial G}{\partial x_i} = \frac{\partial H}{\partial y_i}, \quad \frac{\partial G}{\partial y_i} = -\frac{\partial H}{\partial x_i}$$

- Reinterpret these equations as the equality

$$\{G, \cdot\}_0 = \{H, \cdot\}_1 \quad \{H, \cdot\}_0 = -\{G, \cdot\}_1$$

  with $\{\cdot, \cdot\}_j$ the Poisson brackets associated to the real and imaginary part of the symplectic form $\omega = dz_1 \wedge dz_2$ $(\omega = \omega_0 + i\omega_1)$.
- Check $\{G, H\}_0 = 0$ and $\{H, G\}_1 = 0$ (integrable system).