Metric Diophantine approximation – from continued fractions to fractals

Diophantine approximation is concerned with the quantitative study of the density of the rational numbers inside the real numbers. The Diophantine properties of a real number can be quantified through its approximation properties by rational (and more generally algebraic) numbers. For rational approximation, continued fractions provide an important tool in studying such properties. For higher dimensional problems and for algebraic approximation, different methods are needed.

The metric theory of Diophantine approximation is concerned with the size of sets of numbers enjoying specified Diophantine properties. It is a general feature of the theory that most natural properties give rise to zero–one laws: the set of numbers enjoying the property in question is either null or full with respect to the Lebesgue measure. A more refined study of the null sets can be done using the notions of Hausdorff measure and dimension.

Over the years, considerable work has gone into studying metric Diophantine approximation on subsets of $\mathbb{R}^n$. The initial focus was on curves, surfaces and manifolds, but in recent years much effort has also gone into the study of fractal subsets. Already in the setting of rational approximation of real numbers, many problems which seem simple enough remain open. For instance, it is not known whether the Cantor middle third set contains an algebraic, irrational number (it is conjectured not to do so).

In these lectures, starting from the classical setup, I will work towards the study of metric Diophantine approximation on fractal sets. Along the way, we will touch upon some major open problems in Diophantine approximation, such as the Littlewood conjecture and the Duffin–Schaeffer conjecture. The required elements from fractal geometry will also be covered.

Prerequisites: Basic measure theory is assumed to be prior knowledge. Participants may find it useful to have a look in Cassels: "An introduction to Diophantine approximation".