Distance, energy, and dissipation in gradient flow systems

Maria G. Westdickenberg
RWTH Aachen University

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Abstract

This is an outline and reference list from a course taught in the Winter School at the University of Würzburg in February, 2014.

1 Overview and connection between energy-energy-dissipation and convergence rates in time

Here we considered the effect of a relationship of the form $E^\alpha \lesssim D$, where E is energy, D is dissipation, and $\alpha \in [1, 2)$. We discussed connections with the Lojasiewicz inequality and references include [Lo, S, J, HJ] and related works.

2 Coarsening in the one-d Allen-Cahn equation

Here we discussed the use of an energy-dissipation relationship in our joint work with Otto [OR], where sufficient conditions are given for a gradient flow to display so-called dynamic metastability. We emphasize starting with initial data that is order-one away from the slow manifold and avoiding the use of maximum principles. In [OR] this framework is applied to the one-dimensional Allen-Cahn equation. This equation has been previously studied by many authors, including the seminal work of [CP, FH], the gradient-flow-based approach of [BK], and the more recent detailed analysis of [C].

3 Algebraic rates of convergence for gradient flows in which the energy is convex

Here we looked at work of Brezis [B] on gradient flows with respect to a convex (but not strictly convex) energy functional. In particular, we saw how algebraic and differential relationships among distance, energy, and dissipation lead to sharp convergence rates of the same quantities.

We remarked briefly on a connection with the work of Otto and Villani [OV] on logarithmic Sobolev inequalities. See also [BGL, OV2].

4 Relaxation in the Cahn-Hilliard equation on the line

We introduce a nonlinear, energy-based method to study the stability of energy minimizers of the one-dimensional Cahn-Hilliard equation on the real line subject to ± 1 boundary conditions at infinity [OW]. The method can be viewed in some sense as an adaptation of the method of Brezis to the mildly nonconvex case. Our result for Cahn-Hilliard is optimal given our assumptions on the initial data and is nonperturbative in the sense that we do not require our initial data to be close to the set of stable states. Previous results on relaxation for the one-d Cahn Hilliard equation include [BKT, CCO01, H07a]. See also [CCO00, CO, H07b, HK] and related works.

The method of [OW] may be of more general interest and has also been used recently by Esselborn [E] to study relaxation to equilibrium in the thin-film equation. Also Nolte [N] considered the energy-energy-dissipation relationships in the context of the Cahn-Hilliard equation on a bounded domain subject to Neumann boundary conditions.

Our method relies on negative norms, and we remarked briefly that negative norms have been useful in other contexts. References include [Le, GT, GuW, SS].

5 Energy barrier in the Cahn-Hilliard equation on the torus, $d \geq 2$

Here we return to the static features of an energy landscape in the context of the Cahn-Hilliard equation on the d-dimensional torus for $d \geq 2$. The global energy minimizer of such a problem was studied by [BCK, BGLN, CCELM]. In joint work with Gelantalis [GeW], we ask about the so-called energy barrier surrounding the uniform state and develop upper and lower bounds on the energy barrier that match to leading order.

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